ALGEBRAIC HOMOTOPY CLASSES OF RATIONAL FUNCTIONS

by

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Abstract. — We compute the set $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$ of *naive* homotopy classes of scheme endomorphisms of the projective line $\mathbf{P}^1$ over the spectrum of a field. Our result compares well with Morel's computation in [**9**] of the *group* $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1}$ of $\mathbf{A}^1$-homotopy classes of endomorphisms of $\mathbf{P}^1$: the set $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$ admits an *a priori* monoid structure such that the canonical map $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}} \to \left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1}$ is a group completion.

Résumé **(Classes d'homotopie algébrique de fractions rationnelles).** — Nous déterminons l'ensemble $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$ des classes d'homotopie naïve d'endomorphismes de schémas de la droite projective $\mathbf{P}^1$ au-dessus du spectre d'un corps. Notre résultat se compare bien avec le calcul, effectué par Morel dans [**9**], du *groupe* $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1}$ des classes d'homotopie $\mathbf{A}^1$ d'endomorphismes de $\mathbf{P}^1$: l'ensemble $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$ admet *a priori* une structure de monoïde telle que l'application canonique $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}} \to \left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1}$ soit une complétion en groupe.

1. Introduction

The work of Fabien Morel and Vladimir Voevodsky on $\mathbf{A}^1$-homotopy theory [**8, 10**] provides a convenient framework to do algebraic topology in the setting of algebraic geometry. More precisely, for any fixed field k, Morel and Voevodsky defined a convenient category of *spaces*, say $\mathscr{S}$, containing the category of smooth algebraic k-varieties as a full subcategory, which they endowed with a suitable model structure, in the sense of Quillen's homotopical algebra [**12**]. Thus, given two spaces X and Y in $\mathscr{S}$ (resp. two pointed spaces), the set $\{X,Y\}^{\mathbf{A}^1}$ (resp. the set $[X,Y]^{\mathbf{A}^1}$) of $\mathbf{A}^1$-homotopy classes of *unpointed* morphisms (resp. of *pointed* morphisms) from X to Y is well defined and has all the properties an algebraic topologist can expect. However, for concrete X and Y, these sets are in general very hard to compute.

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At the starting point of $\mathbf{A}^1$-homotopy theory is the notion of *naive homotopy*[(1)] between two morphisms in $\mathscr{S}$. Its definition, first introduced by Karoubi and Villamayor [**6**], mimics that of the usual notion of homotopy between two topological maps, replacing the unit interval $[0, 1]$ by its algebraic analogue, the affine line $\mathbf{A}^1$.

Definition 1.1. — Let X and Y be two spaces in $\mathscr{S}$. A naive homotopy is a morphism

$$F : X \times \mathbf{A}^1 \longrightarrow Y \qquad .$$

The restriction $\sigma(F) := F_{|X\times\{0\}}$ is the source of the homotopy and $\tau(F) := F_{|X\times\{1\}}$ is its target. When X and Y have base points, say x_0 and y_0, we say that F is pointed if its restriction to $\{x_0\} \times \mathbf{A}^1$ is constant equal to y_0.

With this notion, one defines the set $\{X, Y\}^{\mathrm{N}}$ (resp. the set $[X, Y]^{\mathrm{N}}$) of unpointed (resp. of pointed) *naive* homotopy classes of morphisms from X to Y as the quotient of the set of unpointed (resp. of pointed) morphisms by the equivalence relation generated by unpointed (resp. by pointed) naive homotopies. These sets are sometimes easier to compute than their $\mathbf{A}^1$ analogues, but they are not very well behaved. There is a canonical map

$$[X, Y]^{\mathrm{N}} \longrightarrow [X, Y]^{\mathbf{A}^1}$$

which in general is far from being a bijection. This article studies a particular example where this map has an interesting behaviour.

Let k be a base field. In this article, we focus on the set of pointed homotopy classes of endomorphisms of the projective line $\mathbf{P}^1$ over $\operatorname{Spec} k$. Here, our convention is to take the base point in $\mathbf{P}^1$ at $\infty := [1 : 0]$. The main result of this article is the following (see theorem 3.7, corollary 3.11 and theorem 3.24 for more precise and explicit statements).

Theorem 1.2. — *The set* $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$ *of pointed naive homotopy classes of endomorphisms of the projective line admits an* a priori *monoid structure, whose law is denoted by* $\oplus^{\mathrm{N}}$. *The canonical map from the monoid* $([\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}})$ *to the group* $([\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}, \oplus^{\mathbf{A}^1})$ *of* $\mathbf{A}^1$*-homotopy classes of endomorphisms of* $\mathbf{P}^1$ *is a group completion.*

Overview. — We summarize now briefly the organization of the paper.

- Section 2 reviews the classical correspondence between scheme endomorphisms of the projective line over $\operatorname{Spec} k$ and rational functions with coefficients in k. We give also an analogous description of pointed naive homotopies of endomorphisms of $\mathbf{P}^1$ in terms of rational functions with coefficients in the polynomial ring $k[T]$.

(1) In [**10**], the authors use the terminology “elementary homotopy”.

- Section 3 is the core of the article. §3.1 describes a natural monoid structure on the scheme of pointed rational functions which induces the monoid structure on $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$ mentioned in theorem 1.2. In §3.2, we review a classical construction, due to Bézout, which associates a non-degenerate symmetric bilinear form to any rational function. This leads to the definition of a homotopy invariant, which is at the center of our study. In §3.3, the main theorem is stated: the previous homotopy invariant distinguishes all the homotopy classes of rational functions. The proof is given in §3.4. Finally, §3.5 compares our result to the actual $\mathbf{A}^1$-homotopy classes, as computed by Morel in [**9**].
- In section 4, we discuss some natural extensions of our main result. We first give an explicit description of the *unpointed* naive homotopy classes of endomorphisms of $\mathbf{P}^1$ in §4.1. Next, in §4.2, we study the composition of endomorphisms of $\mathbf{P}^1$ in terms of our description of $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$. Finally, for every integer $d \geqslant 2$, we compute the set $\left[\mathbf{P}^1, \mathbf{P}^d\right]^{\mathrm{N}}$ of pointed naive homotopy classes of morphisms from $\mathbf{P}^1$ to $\mathbf{P}^d$. Not surprisingly, the case $d \geqslant 2$ is much easier than the case $d = 1$. Our result still compares well with Morel's computation of the group $\left[\mathbf{P}^1, \mathbf{P}^d\right]^{\mathbf{A}^1}$ of $\mathbf{A}^1$-homotopy classes of morphisms from $\mathbf{P}^1$ to $\mathbf{P}^d$.

Three appendices conclude the article.

- In appendix A, we give an elementary computation, due to Ojanguren [**11**], of the set of naive homotopy classes of non-degenerate symmetric matrices. It is based on an elegant use of Hermite inequality for symmetric bilinear forms over the ring $k[T]$.
- Appendix B proves in detail that the canonical map $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}} \longrightarrow [\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}$ is compatible with monoid structures.
- Appendix C reviews the classical correspondence between rational functions and non-degenerate Hankel matrices.

Acknowledgements. — The material presented here constitutes the first part of the author's Ph-D thesis [**4**]. The main result was first announced in the note [**3**] when the base field is of characteristic not 2. I am very much indebted to Jean Lannes for his precious and constant help.

2. Rational functions and naive homotopies

This section reviews the classical correspondence between pointed endomorphisms of the projective line $\mathbf{P}^1$ (with base point ∞) over the spectrum of the field k and pointed rational functions with coefficients in k. We also give a concrete description of pointed naive homotopies of endomorphisms of $\mathbf{P}^1$ in terms of pointed rational functions with coefficients in the polynomial ring $k[T]$.

2.1. Pointed endomorphisms of the projective line. — We first introduce some notation.

Definition 2.1. — For every positive integer n, the scheme $\mathcal{F}_n$ of pointed degree n rational functions is the open subscheme of the affine space $\mathbf{A}^{2n} = \mathrm{Spec}\,(k[a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}])$ complementary to

the hypersurface of equation[(2)]

$$\mathrm{res}_{n,n}(X^n + a_{n-1}X^{n-1} + \cdots + a_0, b_{n-1}X^{n-1} + \cdots + b_0) = 0 \qquad .$$

Remark 2.2. — Let R be a ring and n a non-negative integer. By the very definition, an R-point of $\mathcal{F}_n$ is a pair (A, B) of polynomials of $R[X]$, where

- A is monic of degree n,
- B is of degree strictly less than n,
- the scalar $\mathrm{res}_{n,n}(A, B)$ is invertible in R.

Such a point is denoted by $\frac{A}{B}$ and is called a pointed degree n rational function with coefficients in R. In the sequel, it is useful to remark that the above condition $\mathrm{res}_{n,n}(A, B) \in R^\times$ is equivalent to the existence of a (necessarily unique) Bézout relation

$$AU + BV = 1$$

with U and V polynomials in $R[X]$ such that $\deg U \leqslant n-2$ and $\deg V \leqslant n-1$.

The precise correspondence between endomorphisms of $\mathbf{P}^1$ and rational functions is now summarized in the following proposition.

Proposition 2.3. — *The datum of a pointed scheme endomorphism of* $\mathbf{P}^1$ *over* $\mathrm{Spec}\, k$*, say* f*, is equivalent to the datum of an non-negative integer* n *and of an element* $\frac{A}{B} \in \mathcal{F}_n(k)$*.*
The integer n *is called the* degree *of* f *and is denoted* $\deg(f)$*; the scalar* $\mathrm{res}_{n,n}(A, B)$ *is called the* resultant *of* f *and is denoted* $\mathrm{res}(f)$*.*

2.2. Naive homotopies. — Recall from definition 1.1 that a pointed naive homotopy of endomorphisms of $\mathbf{P}^1$, is a scheme morphism

$$F : \mathbf{P}^1 \times \mathbf{A}^1 \longrightarrow \mathbf{P}^1$$

satisfying the appropriate base point condition. The following slight generalization of proposition 2.3 gives a description of pointed naive homotopies in terms of rational functions with coefficients in $k[T]$.

Proposition 2.4. — *The datum of a pointed naive homotopy* $F : \mathbf{P}^1 \times \mathbf{A}^1 \longrightarrow \mathbf{P}^1$ *is equivalent to the datum of a non-negative integer* n *and of an element in* $\mathcal{F}_n(k[T])$*. The source* $\sigma(F)$ *and the target* $\tau(F)$ *of* F *are obtained by evaluating the indeterminate* T *at* 0 *and* 1 *respectively.*

Example 2.5. — Let n be a positive integer and b_0 be a unit in $k^\times$.

(2) The notation $\mathrm{res}_{n,n}(A, B)$ stands for the resultant of the polynomials A and B (considered as polynomials of degree less or equal to n). Our conventions, in particular for the sign in the Bézout formula (3.5) below, are those of Bourbaki, [**1**], §6, n°6, IV.

1. Let $A = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ be a monic degree n polynomial of $k[X]$.
   The element
   $$\frac{X^n + Ta_{n-1}X^{n-1} + \cdots + Ta_0}{b_0} \in \mathcal{F}_n(k[T])$$
   gives a pointed naive homotopy between $\frac{A}{b_0}$ and $\frac{X^n}{b_0}$. In other terms, any polynomial is homotopic to its leading term.
2. Let $B = b_{n-1}X^{n-1} + \cdots + b_0$ be a polynomial of degree $\leqslant n-1$ such that $B(0) = b_0$. Then $\frac{X^n}{B}$ is a k-point of $\mathcal{F}_n$ and the element
   $$\frac{X^n}{Tb_{n-1}X^{n-1} + \cdots + Tb_1X + b_0} \in \mathcal{F}_n(k[T])$$
   gives a pointed naive homotopy between $\frac{X^n}{B}$ and $\frac{X^n}{b_0}$.

The simplicity of the above examples of homotopies is somewhat atypical. In general, given a random rational function, it is not *a priori* easy to find non-trivial homotopies. In §3.1, we give a way to produce some such families (in fact just enough to be able to determine $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$), see remark 3.2 (2).

Definition 2.6. — Let f and g be two pointed rational functions over k. We say that f and g are in the same pointed naive homotopy class, and we write $f \overset{\mathrm{p}}{\sim} g$, if there exists a finite sequence of pointed homotopies, say (F_i) with $0 \leqslant i \leqslant N$, such that

- $\sigma(F_0) = f$ and $\tau(F_N) = g$;
- for every $0 \leqslant i \leqslant N-1$, we have $\tau(F_i) = \sigma(F_{i+1})$.

The set of pointed naive homotopy classes $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$ identifies by definition with the quotient of the set $\coprod_{n\geqslant 0} \mathcal{F}_n(k)$ of pointed rational functions with coefficients in k by the equivalence relation $\overset{\mathrm{p}}{\sim}$.

Note that proposition 2.4 implies that two pointed rational functions which are in the same pointed naive homotopy class have same degree and same resultant. In particular, the set $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$ splits as the disjoint union of its components of a given degree
$$\left[\mathbf{P}^1, \mathbf{P}^1\right]^{\mathrm{N}} = \coprod_{n\geqslant 0} \left[\mathbf{P}^1, \mathbf{P}^1\right]^{\mathrm{N}}_n \qquad .$$

Remark 2.7. — Pointed naive homotopies of rational functions are algebraic paths in the scheme of pointed rational functions. It is convenient to reformulate the preceding discussion in terms of the "naive connected components" of this scheme.
Let $\mathcal{G} : \mathrm{Alg}_k \longrightarrow \mathscr{S}et$ be a functor from the category of k-algebras to that of sets. Recall that the naive connected components of $\mathcal{G}$ is the new functor $\pi_0^{\mathrm{N}}\mathcal{G} : \mathrm{Alg}_k \longrightarrow \mathscr{S}et$ which assigns to any k-algebra R the coequalizer of the double-arrow
$$\mathcal{G}(R[T]) \rightrightarrows \mathcal{G}(R)$$

given by evaluation at $T=0$ and $T=1$. Moreover, any natural transformation $\mathcal{T}:\mathcal{G}\longrightarrow\mathcal{H}$ between two such functors induces a natural transformation $\pi_0^{\mathrm{N}}\mathcal{T}:\pi_0^{\mathrm{N}}\mathcal{G}\longrightarrow\pi_0^{\mathrm{N}}\mathcal{H}$.
For every non-negative integer n, proposition 2.4 gives a bijection

$$\left[\mathbf{P}^1,\mathbf{P}^1\right]_n^{\mathrm{N}}\simeq(\pi_0^{\mathrm{N}}\mathcal{F}_n)(k)\qquad.$$

Any scheme morphism $\mathcal{F}_n\longrightarrow X$ produces a homotopy invariant $(\pi_0^{\mathrm{N}}\mathcal{F}_n)(k)\longrightarrow(\pi_0^{\mathrm{N}}X)(k)$.

3. Homotopy classes of rational functions

This section is the core of the article. We first endow the set $[\mathbf{P}^1,\mathbf{P}^1]^{\mathrm{N}}$ with an *a priori* monoid structure in §3.1. We then review in §3.2 a classical construction due to Bézout which associates to any rational function a non-degenerate symmetric matrix called the *Bézout form*. This leads us to the definition of a homotopy invariant. The main result, theorem 3.7, is stated in §3.3 and proved in §3.4. It shows that the homotopy invariant associated to the Bézout form distinguishes all the pointed naive homotopy classes of rational functions. §3.5 finally concludes the section by comparing our naive result to the actual $\mathbf{A}^1$-computation, due to Morel. The result is that the set $[\mathbf{P}^1,\mathbf{P}^1]^{\mathbf{A}^1}$ has for formal reasons a group structure and it turns out that this group is isomorphic to the group completion of the monoid $([\mathbf{P}^1,\mathbf{P}^1]^{\mathrm{N}},\oplus^{\mathrm{N}})$.

3.1. Additions of rational functions. — An important feature in the statement of our results, is the existence of a monoid structure exhibited *a priori* on the set of naive homotopy classes $[\mathbf{P}^1,\mathbf{P}^1]^{\mathrm{N}}$. In fact, there is even a graded monoid structure on the disjoint union scheme

$$\mathcal{F}:=\coprod_{n\geqslant 0}\mathcal{F}_n\qquad.$$

The datum of such a structure is equivalent to the datum of a family of morphisms $\mathcal{F}_{n_1}\times\mathcal{F}_{n_2}\longrightarrow\mathcal{F}_{n_1+n_2}$ indexed by pairs (n_1,n_2) of non-negative integers and subject to an associativity condition.

Let (n_1,n_2) be a pair of non-negative integers. We now describe the above structural morphism on the level of functors of points, that is as a natural transformation of functors from the category of k-algebras to the category of sets:

$$\mathcal{F}_{n_1}(-)\times\mathcal{F}_{n_2}(-)\longrightarrow\mathcal{F}_{n_1+n_2}(-)\qquad.$$

Let R be a k-algebra. Two rational functions $\frac{A_i}{B_i}\in\mathcal{F}_{n_i}(R)$, for $i=1,2$, uniquely define two pairs (U_i,V_i) of polynomials of $R[X]$ with $\deg U_i\leqslant n_i-2$ and $\deg V_i\leqslant n_i-1$ and satisfying Bézout identities $A_iU_i+B_iV_i=1$ (see remark 2.2). Define polynomials A_3,B_3,U_3 and V_3 by setting[(3)]

$$\begin{bmatrix}A_3 & -V_3\\ B_3 & U_3\end{bmatrix}:=\begin{bmatrix}A_1 & -V_1\\ B_1 & U_1\end{bmatrix}\cdot\begin{bmatrix}A_2 & -V_2\\ B_2 & U_2\end{bmatrix}\qquad.$$

(3) The dot in the right-hand term stands for the usual matrix multiplication.

The matrices $\begin{bmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{bmatrix}$ and $\begin{bmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{bmatrix}$ belong to $\mathbf{SL}_2(k[T])$, thus this is also the case for $\begin{bmatrix} A_3 & -V_3 \\ B_3 & U_3 \end{bmatrix}$. This means that we have also a Bézout relation for the polynomials A_3 and B_3. Moreover, observe that $A_3 = A_1A_2 - V_1B_2$ is monic of degree $n_1 + n_2$ and that $B_3 = B_1A_2 + U_1B_2$ is of degree strictly less than $n_1 + n_2$. So $\frac{A_3}{B_3}$ is an R-point of $\mathcal{F}_{n_1+n_2}$. We write

$$\frac{A_1}{B_1} \oplus^{\mathrm{N}} \frac{A_2}{B_2} := \frac{A_3}{B_3} \qquad .$$

Proposition 3.1. — *Let* $\mathcal{F} = \coprod_{n \geqslant 0} \mathcal{F}_n$ *be the scheme of pointed rational functions. Then the above morphisms define a graded monoid structure on* $\mathcal{F}$*:*

$$\oplus^{\mathrm{N}} : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \qquad .$$

Proof. — The only thing to prove is that the associativity condition is satisfied. It is a consequence of the associativity of matrix multiplication. □

Remarks 3.2. —
1. The above monoid structure on $\mathcal{F}$ induces a graded monoid structure on its connected components $(\pi_0^{\mathrm{N}}\mathcal{F})(k) := \coprod_{n \geqslant 0} (\pi_0^{\mathrm{N}}\mathcal{F}_n)(k)$, and thus on $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$. The monoid law on these sets is again denoted by $\oplus^{\mathrm{N}}$.
2. Taking the $\oplus^{\mathrm{N}}$-sum of "trivial" homotopies of rational functions produces "non trivial" homotopies.
3. *Warning*: we use additive notations for the monoid law on $\mathcal{F}$, but we would like to stress that it is *non commutative*. However, we will see in corollary 3.8 that the monoid $([\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}})$ is abelian.

Example 3.3. — We now give some examples of $\oplus^{\mathrm{N}}$-sums of rational functions.

1. One has
$$X \oplus^{\mathrm{N}} X = \frac{X^2 - 1}{X}$$
2. For any pointed rational function $\frac{A}{B}$, one has
$$X \oplus^{\mathrm{N}} \frac{A}{B} = \frac{AX - B}{A} \quad \text{et} \quad \frac{A}{B} \oplus^{\mathrm{N}} \frac{X}{1} = \frac{AX - V}{BX + U} \qquad .$$
3. More generally, for every monic polynomial $P \in k[X]$, and for every unit $b_0 \in k^\times$, one has
$$\frac{P}{b_0} \oplus^{\mathrm{N}} \frac{A}{B} = \frac{AP - \frac{B}{b_0}}{b_0 A} = \frac{P}{b_0} - \frac{1}{b_0^2 \frac{A}{B}} \qquad .$$

Remark 3.4. — When dealing with rational functions with coefficients in the field k, here is a convenient way to understand the preceding addition law $\oplus^{\mathrm{N}}$ on the level of continued fraction expansions.

Every pointed rational fraction $f = \frac{A}{B} \in \mathcal{F}_n(k)$ admits a unique *twisted* continued fraction expansion of the following form:

$$\frac{A}{B} = \frac{P_0}{b_0} - \frac{1}{b_0^2\left(\frac{P_1}{b_1} - \frac{1}{b_1^2(\dots)}\right)} \qquad ,$$

where for each i, $P_i \in k[X]$ is a monic polynomial of positive degree and b_i is a non-zero scalar in $k^\times$. Such an expansion always stops, as the sum of the degrees of the P_i equals the degree of A.
Example 3.3 then shows that the twisted continued fraction expansion of the $\oplus^{\mathrm{N}}$-sum of two rational functions is the concatenation of the two twisted continued fraction expansions.
For the sequel, it is useful to note that every rational function $f \in \mathcal{F}_n(k)$ is tautologically the $\oplus^{\mathrm{N}}$-sum of the polynomials appearing in its twisted continued fraction expansion:

$$f = \frac{P_0}{b_0} \oplus^{\mathrm{N}} \frac{P_1}{b_1} \oplus^{\mathrm{N}} \cdots \oplus^{\mathrm{N}} \frac{P_r}{b_r} \qquad .$$

3.2. The Bézout form. — In the 18^{th} century, Bézout described a way to associate to every rational function a non-degenerate symmetric matrix[(4)]. In modern terms, Bézout's construction describes, for every positive integer n, a scheme morphism

$$\mathrm{Béz}_n : \mathcal{F}_n \longrightarrow \mathcal{S}_n \qquad ,$$

where $\mathcal{S}_n$ is the scheme of non-degenerate $n \times n$ symmetric matrices. The homotopy invariants $(\pi_0^{\mathrm{N}}\mathcal{F}_n)(k) \xrightarrow{\pi_0^{\mathrm{N}}\mathrm{Béz}_n} (\pi_0^{\mathrm{N}}\mathcal{S}_n)(k)$ associated to these morphisms (see remark 2.7) are at the center of our study.

We start first by reviewing Bézout's construction.

Definition 3.5. — Let R be a ring, n be a positive integer and $f = \frac{A}{B}$ be an element of $\mathcal{F}_n(R)$. The polynomial $A(X)B(Y) - A(Y)B(X) \in R[X,Y]$ is divisible by $X - Y$. Let

$$\delta_{A,B}(X,Y) := \frac{A(X)B(Y) - A(Y)B(X)}{X - Y} =: \sum_{1 \leqslant p,q \leqslant n} c_{p,q} X^{p-1} Y^{q-1} \qquad .$$

Observe that the coefficients of $\delta_{A,B}(X,Y)$ are symmetric in the sense that one has

$$c_{p,q} = c_{q,p} \qquad \forall\ 1 \leqslant p,q \leqslant n.$$

The Bézout form of f is the symmetric bilinear form over R^n whose Gram matrix is the $n{\times}n$-symmetric matrix $[c_{p,q}]_{1 \leqslant p,q \leqslant n}$. We denote it $\mathrm{Béz}_n(A,B)$ or $\mathrm{Béz}_n(f)$ and Bézout's formula:

$$\det \mathrm{Béz}_n(f) = (-1)^{\frac{n(n-1)}{2}} \mathrm{res}(f) \tag{3.5}$$

(4) It is sometimes referred as "the bezoutian" of the rational function, but we prefer the terminology "Bézout form".

shows that this form is non-degenerate.
The above construction describes for every positive integer a natural transformation of functors $\mathcal{F}_n(-) \longrightarrow \mathcal{S}_n(-)$ and thus a morphism of schemes

$$\mathrm{B\acute{e}z}_n : \mathcal{F}_n \longrightarrow \mathcal{S}_n \qquad .$$

Remark 3.6. — Following [**5**], chapter III, example 4.8, we give now a more conceptual definition of the Bézout form in terms of Serre duality.
Let $\frac{A}{B}$ be a pointed degree n rational function. Consider the complex of vector bundles over $\mathbf{P}^1$:

$$\mathcal{O}(-n-1) \xrightarrow{\begin{bmatrix} -B \\ A \end{bmatrix}} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \xrightarrow{\begin{bmatrix} A & B \end{bmatrix}} \mathcal{O}(n-1) \qquad .$$

The associated hypercohomology spectral sequence has only one non-trivial differential, $\mathrm{d}_2^{0,1}$: $\mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(-n-1)) \longrightarrow \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(n-1))$. The composite

$$\left(\mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(n-1))\right)^* \simeq \mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(-n-1)) \longrightarrow \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(n-1))$$

identifies with the Bézout form.

3.3. The main theorem. — We can now state our main result.

Theorem 3.7. — *The Bézout invariants distinguish all the naive homotopy classes of rational functions: the map*

$$\Big(\coprod_{n\geqslant 0}(\pi_0^{\mathrm{N}}\mathcal{F}_n)(k), \oplus^{\mathrm{N}}\Big) \xrightarrow{\coprod_{n\geqslant 0} \pi_0^{\mathrm{N}}\mathrm{B\acute{e}z}_n} \Big(\coprod_{n\geqslant 0}(\pi_0^{\mathrm{N}}\mathcal{S}_n)(k), \oplus\Big)$$

is an isomorphism of graded monoids.

Corollary 3.8. — *Since the monoid* $\Big(\coprod_{n\geqslant 0}(\pi_0^{\mathrm{N}}\mathcal{S}_n)(k), \oplus\Big)$ *is abelian,* $\Big([\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}}\Big)$ *is also abelian.*

In order to make the theorem more explicit, we now describe the set of values $(\pi_0^{\mathrm{N}}\mathcal{S}_n)(k)$ of the Bézout invariants. To do so, we need the following definition.

Definition 3.9. —
1. The Witt monoid of the field k is the monoid, for the orthogonal sum $\oplus$, of isomorphism classes of non-degenerate symmetric k-bilinear forms. We denote it $\mathrm{MW}(k)$.
2. Let $\mathrm{MW}^{\mathrm{s}}(k)$ be the monoid of *stable*[(5)] isomorphism classes of non-degenerate symmetric k-bilinear forms. By definition, it is the quotient of $\mathrm{MW}(k)$ where two forms b and b' are to be identified if there exists a form b'' such that $b \oplus b'' \simeq b' \oplus b''$. It comes with a natural grading induced by the rank, and for every positive integer n, we denote $\mathrm{MW}^{\mathrm{s}}_n(k)$ the degree n-component of $\mathrm{MW}^{\mathrm{s}}(k)$.

(5) If the field k is not of characteristic 2, $\mathrm{MW}^{\mathrm{s}}(k)$ is equal to $\mathrm{MW}(k)$. So a reader not interested in the case of a base field of characteristic 2 can forget about stabilization.

3. The Grothendieck-Witt group $\mathrm{GW}(k)$ is the Grothendieck group of the monoid $\mathrm{MW}(k)$ (or of $\mathrm{MW}^{\mathrm{s}}(k)$).

The description of the sets $(\pi_0^{\mathrm{N}}\mathcal{S}_n)(k)$ is given next.

Proposition 3.10. — *Let n be a positive integer.*

1. *The canonical quotient map $q_n : \mathcal{S}_n(k) \longrightarrow \mathrm{MW}^{\mathrm{s}}_n(k)$ factors through $(\pi_0^{\mathrm{N}}\mathcal{S}_n)(k)$:*

$$\begin{tikzcd} \mathcal{S}_n(k) \arrow[r, "q_n"] \arrow[d] & \mathrm{MW}^{\mathrm{s}}_n(k) \\ (\pi_0^{\mathrm{N}}\mathcal{S}_n)(k) \arrow[ur, "\overline{q}_n"'] & \end{tikzcd}$$

2. *Let*

$$\mathrm{MW}^{\mathrm{s}}_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times$$

be the canonical fibre product induced by the discriminant *map* $\mathrm{MW}^{\mathrm{s}}_n(k) \longrightarrow k^\times/k^{\times 2}$. *Then the map*

$$\Big(\coprod_{n\geqslant 0}(\pi_0^{\mathrm{N}}\mathcal{S}_n)(k), \oplus\Big) \xrightarrow{\coprod_{n\geqslant 0}\overline{q}_n\times\det} \Big(\coprod_{n\geqslant 0}\mathrm{MW}^{\mathrm{s}}_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times, \oplus\Big) \qquad .$$

is a monoid isomorphism. (Above, the right-hand term is endowed with the canonical monoid structure induced by the orthogonal sum in $\mathrm{MW}^{\mathrm{s}}(k)$ and the product in $k^\times$).

Proposition 3.10 above is certainly well-known to specialists. An elementary proof, due to Ojanguren [**11**], is postponed until appendix A, because it would be too digressive here. The conjunction of theorem 3.7 and proposition 3.10 gives the following more explicit description of $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$.

Corollary 3.11. — *There is a canonical isomorphism of graded monoids:*

$$\Big(\big[\mathbf{P}^1, \mathbf{P}^1\big]^{\mathrm{N}}, \oplus^{\mathrm{N}}\Big) \simeq \Big(\coprod_{n\geqslant 0}\mathrm{MW}^{\mathrm{s}}_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times, \oplus\Big) \qquad .$$

Example 3.12. — 1. When k is algebraically closed, we have an isomorphism of monoids

$$\big[\mathbf{P}^1, \mathbf{P}^1\big]^{\mathrm{N}} \xrightarrow[\deg\times\ \mathrm{res}]{\simeq} \mathbf{N}\times k^\times \qquad .$$

The Bézout invariant is just the conjunction of the resultant and degree invariants.

2. When k is the field of real numbers $\mathbf{R}$, we have an isomorphism of monoids:

$$\big[\mathbf{P}^1, \mathbf{P}^1\big]^{\mathrm{N}} \xrightarrow[(\mathrm{sign}\circ\mathrm{Béz})\times\ \mathrm{res}]{\simeq} (\mathbf{N}\times\mathbf{N})\times\mathbf{R}^\times \qquad ,$$

sign denoting the signature of a real symmetric bilinear form. In this case, the Bézout invariant is sharper than the resultant and the degree invariants.

3.4. Proof of theorem 3.7. — Three things are to be proved: injectivity, surjectivity and compatibility with the monoid structures of the Bézout map. Each one has an elementary proof. In §3.4.1, we start by proving the surjectivity and the compatibility condition, both at the same time. We then remark in §3.4.2 that injectivity reduces to that of $\mathrm{Béz}_2$. We conclude in §3.4.3 by an independent analysis of the scheme $\mathcal{F}_2$ and of the map $\mathrm{Béz}_2$.

3.4.1. Surjectivity and monoidal compatibility of the Bézout map. — It is very convenient in the sequel to adopt the following conventions.

Definition 3.13. — Let n be a positive integer. For every list of units $u_1, \dots, u_n \in k^\times$, let

- $\langle u_1, \dots, u_n \rangle$ denote the diagonal symmetric bilinear form $\langle u_1 \rangle \oplus \dots \oplus \langle u_n \rangle \in \mathcal{S}_n(k)$.
- $[u_1, \dots, u_n]$ denote the pointed rational function $\frac{X}{u_1} \oplus^{\mathrm{N}} \dots \oplus^{\mathrm{N}} \frac{X}{u_n} \in \mathcal{F}_n(k)$.

The following lemma shows that, up to naive homotopy, any symmetric bilinear form and any rational function is of the preceding form.

Lemma 3.14. — *Let n be a positive integer. Then:*

1. *For any symmetric bilinear form $S \in \mathcal{S}_n(k)$ there exists units $u_1, \dots, u_n \in k^\times$ such that S is homotopic to the diagonal form $\langle u_1, \dots, u_n \rangle$.*
2. *For any pointed rational function $f \in \mathcal{F}_n(k)$ there exists units $u_1, \dots, u_n \in k^\times$ such that we have*
$$f \overset{\mathrm{p}}{\sim} [u_1, \dots, u_n] \qquad .$$

Proof. —
1. Suppose first that the characteristic of the field k is not 2. Then every symmetric matrix $S \in \mathcal{S}_n(k)$ is conjugate by an element $P \in \mathbf{SL}_n(k)$ to a diagonal matrix. Decomposing P into a product of elementary matrices and multiplying by $1 - T$ the non diagonal elements of these elementary matrices yields a homotopy (that is to say an element of $\mathcal{S}_n(k[T])$) to a diagonal matrix.
If k is of characteristic 2, then S is conjugate by an element $P \in \mathbf{SL}_n(k)$ to a block diagonal matrix, with possible $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ terms. In addition to the preceding argument, one can use the homotopy $\begin{bmatrix} T & 1 \\ 1 & 0 \end{bmatrix}$ to link S to a diagonal matrix.
2. We prove this point by induction on the degree n of f.
As noticed in remark 3.4, a rational function $f \in \mathcal{F}_n(k)$ is tautologically the $\oplus^{\mathrm{N}}$-sum of some polynomials, say $P_1 \oplus^{\mathrm{N}} \dots \oplus^{\mathrm{N}} P_k$. Thus one can assume that f is a polynomial. Example 2.5 (1) then shows that a polynomial is always homotopic to its leading term. So it's enough to treat the case of a monomial $\frac{X^n}{u}$, with $u \in k^\times$. Now, example 2.5 (2) shows that the element $\frac{X^n}{TX^{n-1}+u} \in \mathcal{F}_n(k[T])$ defines a homotopy between $\frac{X^n}{u}$ and $\frac{X^n}{X^{n-1}+u}$. But this last rational function decomposes as $X \oplus^{\mathrm{N}} g$ for some $g \in \mathcal{F}_{n-1}(k)$. One concludes by using the inductive hypothesis on g.

$\square$

The monoids $((\pi_0^{\mathrm{N}}\mathcal{S})(k), \oplus)$ and $((\pi_0^{\mathrm{N}}\mathcal{F})(k), \oplus^{\mathrm{N}})$ are generated by their degree 1 components. Since the map $\pi_0^{\mathrm{N}}\mathrm{Béz}_1 : (\pi_0^{\mathrm{N}}\mathcal{F}_1)(k) \longrightarrow (\pi_0^{\mathrm{N}}\mathcal{S}_1)(k)$ is a bijection, the following lemma shows that the Bézout form of a rational function of the form $[u_1, \dots, u_n] \in \mathcal{F}_n(k)$ is homotopic to the diagonal form $\langle u_1, \dots, u_n\rangle$.

Lemma 3.15. — *Let $\frac{A}{B} \in \mathcal{F}_n(k)$ and $u \in k^\times$. Then the Bézout form of $\frac{X}{u} \oplus^{\mathrm{N}} \frac{A}{B}$ is conjugate*[6] *by an element in $\mathbf{SL}_{n+1}(k)$ to the block diagonal form $\langle u\rangle \oplus \mathrm{Béz}_n(A, B)$.*

Proof. — By definition, one has $\frac{X}{u} \oplus^{\mathrm{N}} \frac{A}{B} = \frac{XA - \frac{B}{u}}{uA}$. Using the notations introduced in definition 3.5, we have

$$\delta_{XA-\frac{B}{u}, uA}(X, Y) = uA(X)A(Y) + \delta_{A,B}(X, Y) \qquad .$$

In the basis $(1, X, \dots, X^{n-1}, A(X))$, the matrix of the Bézout form is diagonal as announced. $\square$

This proves that the Bézout map induces a surjective morphism of monoids.

3.4.2. Injectivity. — Let n be a positive integer. To prove the injectivity of the map $\pi_0^{\mathrm{N}}\mathrm{Béz}_n$, we prove the injectivity of the composite

$$(\pi_0^{\mathrm{N}}\mathcal{F}_n)(k) \xrightarrow{\pi_0^{\mathrm{N}}\mathrm{Béz}_n} (\pi_0^{\mathrm{N}}\mathcal{S}_n)(k) \xrightarrow{\overline{q}_n \times \det} \mathrm{MW}^{\mathrm{s}}_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times \qquad .$$

Because up to homotopy any rational function is a $\oplus^{\mathrm{N}}$-sum of degree 1 monomials (*c.f.* lemma 3.14), the injectivity of the previous map can be reformulated as follows.

Proposition 3.16. — *Let $u_1, \dots, u_n, v_1, \dots, v_n$ be a list of units in $k^\times$. If the classes in $\mathrm{MW}^{\mathrm{s}}_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times$ of the diagonal forms $\langle u_1, \dots, u_n\rangle$ and $\langle v_1, \dots, v_n\rangle$ are equal, then $[u_1, \dots, u_n] \overset{\mathrm{p}}{\sim} [v_1, \dots, v_n]$ holds in $\mathcal{F}_n(k)$.*

Proof. — Let's first introduce some *ad hoc* terminology.
Two diagonal forms $\langle u_1, \dots, u_n\rangle$ and $\langle v_1, \dots, v_n\rangle$ are said to be equivalent through an *elementary* $\mathbf{SL}_2(k)$*-transformation* if there exists an subscript $1 \leqslant i \leqslant n-1$ such that:

- the 2-forms $\langle u_i, u_{i+1}\rangle$ and $\langle v_i, v_{i+1}\rangle$ are $\mathbf{SL}_2(k)$-equivalent;
- for all $j \neq i, i+1$, we have $u_j = v_j$.

The next lemma is a slight reformulation of [**7**], chapter III, lemma 5.6 which is gives a presentation of the Witt group $\mathrm{W}(k)$ by generators and relations.

Lemma 3.17. — *Let $\langle u_1, \dots, u_n\rangle$ and $\langle v_1, \dots, v_n\rangle$ be two diagonal forms in $\mathcal{S}_n(k)$. Their images in $\mathrm{MW}^{\mathrm{s}}_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times$ are equal if and only if one can "pass from one to the other" by a finite sequence of elementary $\mathbf{SL}_2(k)$-transformations.*

[6] And is thus also homotopic!

So, in order to prove proposition 3.16, it's enough to deal with the case $n = 2$. This case is analysed independently in the next paragraph.

3.4.3. Rational functions of degree 2. — Let $\mathbf{G_a}$ be the "additive group", that is to say the affine line $\mathbf{A}^1$ seen as a group scheme. For every positive integer n, $\mathbf{G_a}$ acts freely on $\mathcal{F}_n$ by translations, *i.e.* by the formula

$$h \cdot \frac{A}{B} := \frac{A + hB}{B} \qquad ,$$

on the level of points. The following lemma shows that the $\mathbf{G_a}$-torsor $\mathcal{F}_n$ splits.

Lemma 3.18. — *Let R be a ring and $\frac{A}{B}$ be an element of $\mathcal{F}_n(R)$. There exists a unique pair of polynomials (U_1, V_1) of $R[X]$ with $\deg(U_1) = n-1$, $\deg(V_1) \leqslant n-1$ and such that $AU_1 + BV_1 = X^{2n-1}$. Let $\varphi_n\left(\frac{A}{B}\right)$ be the opposite of the coefficient of X^{n-1} in V_1. Then the associated scheme morphism*

$$\varphi_n : \mathcal{F}_n \longrightarrow \mathbf{A}^1$$

is $\mathbf{G_a}$-equivariant. In particular, $\mathcal{F}_n$ splits as the product $\varphi_n^{-1}(0) \times \mathbf{A}^1$.

Proof. — Let A, B, U_1 and V_1 be like above. If one changes (A, B) to $(A + hB, B)$, then (U_1, V_1) is changed to $(U_1, V_1 - hU_1)$. The claim follows since U_1 is necessarily monic. □

Moreover, observe that the morphism $\mathrm{Béz}_n : \mathcal{F}_n \longrightarrow \mathcal{S}_n$ is by construction $\mathbf{G_a}$-equivariant when $\mathcal{S}_n$ is endowed with the trivial action. In dimension 2, the morphism $\mathrm{Béz}_2$ induces an isomorphism between $\varphi_2^{-1}(0)$ and $\mathcal{S}_2$.

Proposition 3.19. — *The morphism*

$$\mathcal{F}_2 \xrightarrow{\mathrm{Béz}_2 \times \varphi_2} \mathcal{S}_2 \times \mathbf{A}^1$$

is a $\mathbf{G_a}$-equivariant isomorphism of schemes.

Corollary 3.20. — *The map*

$$(\pi_0^{\mathrm{N}}\mathcal{F}_2)(k) \xrightarrow{\pi_0^{\mathrm{N}}\mathrm{Béz}_2} (\pi_0^{\mathrm{N}}\mathcal{S}_2)(k) \xrightarrow{\overline{q}_2 \times \det} \mathrm{MW}_2^{\mathrm{s}}(k) \underset{k^\times/k^{\times 2}}{\times} k^\times$$

is injective.

Corollary 3.20 concludes the proof of proposition 3.16.

Proof of proposition 3.19. — One can write down the inverse morphism $\psi : \mathcal{S}_2 \longrightarrow \varphi_2^{-1}(0)$ by solving a system of two equations with two unknowns. The formula is

$$\psi\left(\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\right) = \frac{X^2 + \frac{\alpha\beta}{\beta^2 - \alpha\gamma}X + \frac{\alpha^2}{\beta^2 - \alpha\gamma}}{\gamma X + \beta} \qquad .$$

□

This proof of theorem 3.7 is now complete. □

Remark 3.21. — For $n \geqslant 3$, proposition 3.19 generalizes to an isomorphism $\mathcal{F}_n \simeq \mathcal{H}_n \times \mathbf{A}^1$, where $\mathcal{H}_n$ is the scheme of non-degenerate $n \times n$ *Hankel matrices*, see appendix C. The miracle that allows our computation is that non-degenerate 2×2 Hankel matrices identify with regular non-degenerate symmetric matrices in which naive homotopies are well understood.

3.5. Comparing naive and motivic homotopy classes. — Recall that $\mathrm{Ho}(\mathscr{S})$ is the homotopy category of spaces of $\mathbf{A}^1$-homotopy theory defined by Morel and Voevodsky in [**10**]. For any space $X \in \mathscr{S}$, let $\mathbf{\Sigma}^s X$ be its "suspension with respect to the simplicial circle", that is to say

$$\mathbf{\Sigma}^s X := \operatorname{hocolim} \left(\begin{tikzcd}[column sep=small] \mathrm{pt} & & \mathrm{pt} \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd} \right) \qquad .$$

Lemma 3.22. — *In the homotopy category* $\mathrm{Ho}(\mathscr{S})$*, there is an equivalence*

$$\mathbf{P}^1 \approx \mathbf{\Sigma}^s \mathbf{G_m} \qquad .$$

Above, $\mathbf{G_m}$ *is the multiplicative group, that is to say the scheme* $\mathbf{A}^1 - \{0\}$*.*

Proof. — This is a consequence of $\mathbf{P}^1$ being covered by two contractible open subschemes (two affine lines $\mathbf{A}^1$) intersecting along a $\mathbf{G_m}$. □

It follows formally from the preceding lemma that the set $[\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}$ is equipped with a *group structure*, whose law is denoted by $\oplus^{\mathbf{A}^1}$. The canonical map $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}} \longrightarrow [\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}$ is thus not a bijection: its reaches only elements of non-negative degrees. However, Morel's computation shows that the error in the naive computation is as small as possible.

Theorem 3.23 **(Morel, [9], theorem 4.36)**. — *There is a group isomorphism*

$$\left([\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}, \oplus^{\mathbf{A}^1} \right) \simeq \left(\mathrm{GW}(k) \underset{k^\times / k^{\times 2}}{\times} k^\times, \oplus \right) \qquad .$$

In particular, the group $\left([\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}, \oplus^{\mathbf{A}^1} \right)$ *is (abstractly) isomorphic to the group completion of the monoid* $\left([\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}} \right)$*.*

As one can expect, the group completion is induced by the canonical map between the monoid of naive homotopy classes to the group of $\mathbf{A}^1$ homotopy classes.

Theorem 3.24. — *The canonical map*

$$\left([\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}} \right) \longrightarrow \left([\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}, \oplus^{\mathbf{A}^1} \right)$$

is a group completion.

Proof. — The main point is to prove the compatibility between the monoid law $\oplus^{\mathrm{N}}$ on $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$ and the group law on $[\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}$. Namely, we have the following

Proposition 3.25. — *The canonical map*

$$\left([\mathbf{P}^1,\mathbf{P}^1]^{\mathrm{N}},\oplus^{\mathrm{N}}\right)\longrightarrow\left([\mathbf{P}^1,\mathbf{P}^1]^{\mathbf{A}^1},\oplus^{\mathbf{A}^1}\right)$$

is a monoid morphism.

A detailed proof of proposition 3.25 is postponed to appendix B.

□

Remark 3.26. — The question of the comparison between the naive addition law $\oplus^{\mathrm{N}}$ and the $\mathbf{A}^1$-addition law $\oplus^{\mathbf{A}^1}$ makes sense on the level of the spaces of morphisms and not only on the level of homotopy classes. In particular, it is natural to ask if for any pair of non-negative integers (m,n) the following diagram homotopy commutes[(7)]:

$$\begin{tikzcd}\mathcal{F}_m\times\mathcal{F}_n \arrow[r, "\oplus^{\mathrm{N}}"] \arrow[d, hook] & \mathcal{F}_{m+n} \arrow[d, hook] \\ \Omega^{\mathbf{P}^1}_m\mathbf{P}^1\times\Omega^{\mathbf{P}^1}_n\mathbf{P}^1 \arrow[r, "\oplus^{\mathbf{A}^1}"] & \Omega^{\mathbf{P}^1}_{m+n}\mathbf{P}^1\end{tikzcd} \quad .$$

4. Related computations

The previous computation has natural extensions which we consider now.

- In §4.1, we compute the set of free homotopy classes of scheme endomorphisms of $\mathbf{P}^1$.
- In §4.2, we make explicit the monoid structure induced on $[\mathbf{P}^1,\mathbf{P}^1]^{\mathrm{N}}$ by the composition of endomorphisms of $\mathbf{P}^1$.
- Finally, in §4.3, we compute the set of naive homotopy classes of morphisms from $\mathbf{P}^1$ to $\mathbf{P}^d$.

4.1. Free homotopy classes of rational functions. — We compute here the set $\{\mathbf{P}^1,\mathbf{P}^1\}^{\mathrm{N}}$ of naive homotopy classes of *unpointed* endomorphisms of $\mathbf{P}^1$. The result is mostly a consequence of our previous computation of $[\mathbf{P}^1,\mathbf{P}^1]^{\mathrm{N}}$.

The description of free endomorphisms of $\mathbf{P}^1$ in terms of rational functions is the following.

Definition 4.1. — For every non-negative integer n, the scheme $\mathscr{U}_n$ of unpointed degree n rational functions is the open subscheme of $\mathbf{P}^{2n+1}:=\mathrm{Proj}\left(k[a_0,\dots,a_n,b_0,b_n]\right)$ complementary to the hypersurface of equation

$$\mathrm{res}_{n,n}(a_nX^n+\cdots+a_0,b_nX^n+\cdots+b_0)=0 \qquad .$$

[(7)]In the diagram, $\Omega^{\mathbf{P}^1}\mathbf{P}^1$ is the space (in $\mathscr{S}$) of pointed morphisms from $\mathbf{P}^1$ to itself. This space splits as the disjoint union of its components of constant degree: $\Omega^{\mathbf{P}^1}\mathbf{P}^1=\coprod_{k\in\mathbf{Z}}\Omega^{\mathbf{P}^1}_k\mathbf{P}^1$. For any non-negative k, $\mathcal{F}_k$ is seen as a subspace of $\Omega^{\mathbf{P}^1}_k\mathbf{P}^1$.

Proposition 4.2. — 1. *The datum of an endomorphism of* $\mathbf{P}^1$ *is equivalent to the datum of a non-negative integer* n *(its* degree*) and of an element in* $\mathscr{U}_n(k)$*.*

2. *Moreover, the datum of a naive homotopy* $\mathbf{P}^1 \times \mathbf{A}^1 \longrightarrow \mathbf{P}^1$ *is equivalent to the datum of a non-negative integer* n *and of an element in* $\mathscr{U}_n(k[T])$*.*

We denote $\overset{\mathrm{u}}{\sim}$ the equivalence relation generated by *unpointed* naive homotopies and by $\{\mathbf{P}^1, \mathbf{P}^1\}^{\mathrm{N}}$ the set of unpointed naive homotopy classes of endomorphisms of $\mathbf{P}^1$. It follows from proposition 4.2 that the degree is still a naive homotopy invariant. Thus the set $\{\mathbf{P}^1, \mathbf{P}^1\}^{\mathrm{N}}$ splits as the disjoint union

$$\{\mathbf{P}^1, \mathbf{P}^1\}^{\mathrm{N}} = \coprod_{n \geqslant 0} \{\mathbf{P}^1, \mathbf{P}^1\}^{\mathrm{N}}_n$$

of its components of fixed degree.

Let n be a positive integer. Note that a k-point in $\mathscr{U}_n$ gives a pair of coprime polynomials (A, B) only up to the multiplication by a unit of k. Thus the Bézout form of an unpointed endomorphism of $\mathbf{P}^1$ is defined only up to multiplication by the square of a unit of k. The result is that the Bézout form leads again to an invariant that distinguishes all the homotopy classes.

Theorem 4.3. — *The canonical map of graded sets:*

$$\begin{array}{rclcl} \{\mathbf{P}^1, \mathbf{P}^1\}^{\mathrm{N}} = & \coprod_{n \geqslant 0} \{\mathbf{P}^1, \mathbf{P}^1\}^{\mathrm{N}}_n & \longrightarrow & \coprod_{n \geqslant 0} \mathrm{MW}^{\mathrm{s}}_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times/k^{\times 2n} \\ & \left[\frac{A}{B}\right] & \mapsto & [\mathrm{B\acute{e}z}(A, B),\ \det \mathrm{B\acute{e}z}(A, B)] \end{array}$$

is a bijection.

Proof of theorem 4.3. — This is a consequence of corollary 3.11 and of the following lemma. □

Lemma 4.4. — 1. *Any free rational function is naively homotopic to a pointed rational function.*

2. *Let f and g be two* pointed *rational functions. Then one has the relation $f \overset{\mathrm{u}}{\sim} g$ if and only if there exists a non-zero element $\lambda \in k^\times$ with $f \overset{\mathrm{p}}{\sim} \lambda^2 g$.*

Proof. — 1. Let $f = \frac{A}{B}$ represent a rational function and let α_1 be a matrix in $\mathbf{SL}_2(k)$ such that $\alpha_1 \cdot \infty = f(\infty)$, for the usual action of $\mathbf{SL}_2(k)$ on $\mathbf{P}^1(k)$. Let $\alpha(T)$ be an algebraic path in $\mathbf{SL}_2(k[T])$ linking the identity to α_1. (Again, this can be obtained using a decomposition of α_1 as a product of elementary matrices). The column vector $\alpha(T)^{-1} \cdot \begin{pmatrix} A \\ B \end{pmatrix}$ is a $k[T]$-point of $\mathscr{U}_n$ and thus yields a homotopy between f and the *pointed* rational function $\alpha_1^{-1} \cdot \frac{A}{B}$.

2. Suppose first that we have two rational functions f, g such that $f \overset{\mathrm{u}}{\sim} g$ and let's show that there exists a unit $\lambda \in k^\times$ such that $f \overset{\mathrm{p}}{\sim} \lambda^2 g$.
Let

$$f = f_0 \underset{F_0(T)}{\overset{\mathrm{u}}{\sim}} f_1 \underset{F_1(T)}{\overset{\mathrm{u}}{\sim}} \cdots \underset{F_{N-1}(T)}{\overset{\mathrm{u}}{\sim}} f_N = g$$

be a sequence of elementary homotopies between f and g. Let $\alpha(T)$ be an matrix in $\mathbf{SL}_2(k[T])$ such that we have $F_0(T,\infty) = \alpha(T)\cdot\infty$ and $\alpha(0) = \mathrm{id}$. The path $\alpha(T)^{-1}\cdot F_0(T)$ yields a pointed homotopy between the pointed rational functions f_0 and $\alpha(1)^{-1}\cdot f_1$. Moreover, for $N > 1$, we then have a sequence of $N-1$ free homotopies

$$\alpha(1)^{-1}\cdot f_1 \underset{\alpha(1-T)^{-1}F_1(T)}{\sim} f_2 \underset{F_2(T)}{\sim} \cdots \underset{F_{N-1}(T)}{\sim} f_N = g \qquad .$$

Thus the result will follow by induction from the case $N = 1$.
When $N = 1$, $\alpha(1)^{-1}$ is of the form $\begin{bmatrix}\lambda & \mu\\ 0 & \lambda^{-1}\end{bmatrix}$. So, $\begin{bmatrix}1 & T\lambda\mu\\ 0 & 1\end{bmatrix}\cdot\alpha(T)^{-1}\cdot F_0(T)$ gives a pointed homotopy between f_0 and $\lambda^2 f_1$.
We now show the converse. For every $\lambda\in k^\times$, a path in $\mathbf{SL}_2(k[T])$ between the identity matrix and $\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}$ yields a free homotopy between any rational function g and $\lambda^2 g$. The result follows. □

4.2. Composition of rational functions. — The set $[\mathbf{P}^1,\mathbf{P}^1]^{\mathrm{N}}$ admits a second monoid structure induced by composition of morphisms. We make explicit this new structure in terms of our previous description of $[\mathbf{P}^1,\mathbf{P}^1]^{\mathrm{N}}$.

Definition 4.5. — Define a new composition law, say $\circ$, on $\coprod_{n\geqslant 0}\mathrm{MW}^{\mathrm{s}}_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times$ by

$$(b_1,\lambda_1)\circ(b_2,\lambda_2) := \left(b_1\otimes b_2, \lambda_1^{\dim b_2}\lambda_2^{(\dim b_1)^2}\right) \qquad .$$

This law is associative and endows $\coprod_{n\geqslant 0}\mathrm{MW}^{\mathrm{s}}_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times$ with a monoid structure.

Theorem 4.6. — *The following map induces an isomorphism of graded "bi-monoids"*

$$\begin{array}{ccc}\left([\mathbf{P}^1,\mathbf{P}^1]^{\mathrm{N}},\oplus^{\mathrm{N}},\circ\right) & \longrightarrow & \left(\coprod_{n\geqslant 0}\mathrm{MW}^{\mathrm{s}}_n(k) \underset{k^\times/k^{\times 2}}{\times} k^\times,\oplus,\circ\right)\\ f & \mapsto & [\mathrm{B\acute{e}z}(f),\mathrm{res}(f)]\end{array}$$

Remark 4.7. — *Warning:* the triple $([\mathbf{P}^1,\mathbf{P}^1]^{\mathrm{N}},\oplus^{\mathrm{N}},\circ)$ is *not* a semi-ring. In general, one has distributivity of $\circ$ over $\oplus^{\mathrm{N}}$ only on the left-hand side. That is to say that for any triple (f,g_1,g_2) of pointed rational functions, we have

$$f\circ(g_1\oplus^{\mathrm{N}} g_2) \overset{\mathrm{p}}{\sim} (f\circ g_1)\oplus^{\mathrm{N}}(f\circ g_2) \qquad ,$$

but in general

$$(g_1\oplus^{\mathrm{N}} g_2)\circ f \overset{\mathrm{p}}{\sim} (g_1\circ f)\oplus^{\mathrm{N}}(g_2\circ f)$$

does not hold.

Proof of theorem 4.6. — Since any pointed rational function is up to homotopy a $\oplus^{\mathrm{N}}$-sum of degree 1 rational functions, theorem 4.6 follows from the following lemma.

Lemma 4.8. — *Let $a \in k^{\times}$ be a unit, m and n be positive integers, and $f = \frac{A}{B} \in \mathcal{F}_m(k)$ and $g = \frac{C}{D} \in F_n(k)$ be two pointed rational functions. Then*

1. *We have $\frac{X}{a} \circ f = \frac{1}{a} f$.*
2. *In the stable Witt monoid $\mathrm{MW}^{\mathrm{s}}_{(m+1)n}(k)$, we have*
$$\mathrm{Béz}_{(m+1)n}\left((X \oplus^{\mathrm{N}} f) \circ g\right) = \mathrm{Béz}_n(g) \oplus^{\mathrm{N}} \mathrm{Béz}_{mn}(f \circ g) \qquad .$$
3. *We have*
$$\det \mathrm{Béz}_{(n+1)m}\left((X \oplus^{\mathrm{N}} f) \circ g\right) = \det \mathrm{Béz}_n(g)^{2m+1} \cdot \det \mathrm{Béz}_{nm}(f \circ g) \qquad .$$

Proof. — 1. This is true by definition.

2. Let $\dfrac{\widetilde{A}}{\widetilde{B}}$ be the pointed rational function representing $f \circ g$. By definition, one has:
$$\widetilde{A}(X) = \sum_{i=0}^{m} a_i C(X)^i D(X)^{m-i} \quad \text{et} \quad \widetilde{B}(X) = \sum_{i=1}^{m} b_i C(X)^i D(X)^{m-i} \qquad .$$
Since we have $X \oplus^{\mathrm{N}} \frac{A}{B} = X - \frac{B}{A}$, the endomorphism $(X \oplus^{\mathrm{N}} f) \circ g$ is represented by the rational function
$$\frac{C}{D} - \frac{\widetilde{B}}{\widetilde{A}} = \frac{C\widetilde{A} - D\widetilde{B}}{D\widetilde{A}} \qquad .$$
Moreover, using the notations of definition 3.5, we have the identity
$$\delta_{C\widetilde{A}-D\widetilde{B},D\widetilde{A}}(X,Y) = \widetilde{A}(X)\widetilde{A}(Y)\delta_{C,D}(X,Y) + D(X)D(Y)\delta_{\widetilde{A},\widetilde{B}}(X,Y) \qquad .$$
Observing the congruence $\widetilde{A} \equiv C^m \mod D$, we deduce
$$\mathrm{res}(\widetilde{A}, D) = \mathrm{res}(C^m, D) = \mathrm{res}(C, D)^m \in k^{\times} \qquad .$$
Thus the family $(D(X), X \cdot D(X), \ldots, X^{mn-1} \cdot D(X), \widetilde{A}(X), X \cdot \widetilde{A}(X), \ldots, X^{m-1}\widetilde{A}(X))$ gives a basis of the k-vector space of polynomial of degree $< (n+1)m - 1$. And in this basis, the matrix of the form $\mathrm{Béz}_{(n+1)m}((X \oplus^{\mathrm{N}} f) \circ g)$ is
$$\begin{bmatrix} \mathrm{Béz}_{mn}(\widetilde{A}, \widetilde{B}) & 0 \\ 0 & \mathrm{Béz}_m(C, D) \end{bmatrix} \qquad .$$

3. This point follows from the proof of the previous one. Indeed, we have just proved the following matrix identity
$$\mathrm{Béz}_{(n+1)m}(C\widetilde{A} - D\widetilde{B}, D\widetilde{A}) = {}^t\mathrm{Syl}(\widetilde{A}, D) \begin{bmatrix} \mathrm{Béz}_{mn}(\widetilde{A}, \widetilde{B}) & 0 \\ 0 & \mathrm{Béz}_m(C, D) \end{bmatrix} \mathrm{Syl}(\widetilde{A}, D) \qquad ,$$
where Syl is the Sylvester matrix (see [**1**], §6, n°6, IV). The result now follows from the relation $\det \mathrm{Syl}(\widetilde{A}, D) = \mathrm{res}(\widetilde{A}, D) = \mathrm{res}(C, D)^m$.

□

This completes the proof of theorem 4.6. □

4.3. Naive homotopy classes of morphisms to higher dimensional projective spaces. — Let $d \geqslant 2$ be an integer. We compute now naive homotopy classes of morphisms from $\mathbf{P}^1$ to $\mathbf{P}^d$ (over $\operatorname{Spec} k$). The group of $\mathbf{A}^1$-homotopy classes of morphisms from $\mathbf{P}^1$ to $\mathbf{P}^d$ is also determined in [**9**] and we can again compare our result to it. Not surprisingly, the computation is much easier than the previous one for $d = 1$.

For brevity, we treat only the case of pointed morphisms. The base point in $\mathbf{P}^d$ is taken at $\infty := [1 : 0 : \ldots : 0]$. Pointed scheme morphisms from $\mathbf{P}^1$ to $\mathbf{P}^d$ and naive homotopies between them admit the following concrete description, analogous to that of proposition 2.3 and proposition 2.4.

Definition 4.9. — For every non-negative integer n, let $\mathcal{F}_n^d(-)$ be the functor from the category of k-algebras to the category of sets which maps any k-algebra R to the set of pairs (A, B), where A is a monic polynomial of $R[X]$ of degree n, and $B := (B_1, \ldots, B_d)$ is a d-tuple of polynomials, each of degree strictly less than n and such that the ideal generated by the family $\{A, B_1, \ldots, B_d\}$ is $R[X]$. These functors $\mathcal{F}_n^d(-)$ are representable by smooth schemes, which we denote $\mathcal{F}_n^d$ again.

Proposition 4.10. — 1. *The datum of a pointed morphism $\mathbf{P}^1 \longrightarrow \mathbf{P}^d$ is equivalent to the datum of a non-negative integer n, called the* the degree *of the morphism, and of an element in $\mathcal{F}_n^d(k)$.*
2. *The datum of a pointed homotopy $\mathbf{P}^1 \times \mathbf{A}^1 \longrightarrow \mathbf{P}^d$ is equivalent to the datum of a non-negative integer n and of an element in $\mathcal{F}_n^d(k[T])$.*

Remark 4.11. — Let R be a ring and A be a monic polynomial in $R[X]$. The datum of a B such that (A, B) is an R-point of $\mathcal{F}_n^d$ is equivalent to the datum of an element

$$(\overline{B_1}, \ldots, \overline{B_d}) \in (\mathbf{A}^d - \{0\})\left({R[X]}/{(A)}\right) \qquad .$$

This point of view leads to the definition of *Atiyah and Hitchin schemes*, see [**2, 4**].

As in definition 2.6, we denote $\overset{\mathrm{p}}{\sim}$ the equivalence relation generated by pointed naive homotopies of morphisms from $\mathbf{P}^1$ to $\mathbf{P}^d$ and by $\left[\mathbf{P}^1, \mathbf{P}^d\right]^{\mathrm{N}}$ the corresponding set of naive homotopy classes. Since the degree of morphisms is invariant through naive homotopies, the set $\left[\mathbf{P}^1, \mathbf{P}^d\right]^{\mathrm{N}}$ splits as the union of its degree components:

$$\left[\mathbf{P}^1, \mathbf{P}^d\right]^{\mathrm{N}} = \coprod_{n \geqslant 0} \left[\mathbf{P}^1, \mathbf{P}^d\right]_n^{\mathrm{N}} \qquad .$$

We can now state the result.

Theorem 4.12. — *For every $d \geqslant 2$, the degree map* deg

$$\left[\mathbf{P}^1, \mathbf{P}^d\right]^{\mathrm{N}} \xrightarrow[\simeq]{\deg} \mathbf{N}$$

is a bijection.

Proof. — Fix a non-negative integer n. We are going to prove that the set $(\pi_0^{\mathrm{N}}\mathcal{F}_n^d)(k)$ contains only one element. More precisely, we are going to link any element $(A, B_1, \dots, B_d) \in \mathcal{F}_n^d(k)$ to $(X^n, 1, \dots, 1)$ by a sequence of pointed naive homotopies.
Notice first, that it's enough to link $(A, B_1, \dots, B_d)$ to $(A, 1, \dots, 1)$ since the pointed homotopy

$$((1-T)A + TX^n, 1, \dots, 1)$$

then links $(A, 1, \dots, 1)$ to $(X^n, 1, \dots, 1)$.
To do so, decompose the polynomial A as a product of irreducible factors:

$$A = \prod_{i=1}^{r} P_i^{r_i} \qquad .$$

As noticed in remark 4.11, the set of k-points of $\mathcal{F}_n^d$ with first coordinate equal to A is in bijection with $(\mathbf{A}^d - \{0\})\left(k[X]/_{(A)}\right)$. The Chinese remainder theorem identifies this set and

$$\prod_{i=1}^{r} (\mathbf{A}^d - \{0\})\left(k[X]/_{(P_i^{r_i})}\right) \qquad .$$

This means that it's enough to treat the case when A is a power of an irreducible polynomial, say $A = P^r$. In this case, the ring $R := k[X]/_{(P^r)}$ is local. A d-tuple $(\overline{B_1}, \dots, \overline{B_d}) \in R^n$ represents an element of $(\mathbf{A}^d - \{0\})\left(k[X]/_{(P^r)}\right)$ if and only if one of the $\overline{B_i}$ is a unit of R. Up to reordering, we can assume that $\overline{B_1}$ is such a unit. The $k[T]$-point of $\mathcal{F}_n^d$

$$(A, B_1, (1-T)B_2 + T, \dots, (1-T)B_d + T)$$

then yields a pointed homotopy linking $(A, B_1, B_2, \dots, B_d)$ to $(A, B_1, 1, \dots, 1)$. One concludes using the pointed homotopy $(A, (1-T)B_1 + T, 1, \dots, 1)$ from $(A, B_1, 1, \dots, 1)$ to $(A, 1, 1, \dots, 1)$. $\square$

Remark 4.13. — It follows from theorem 4.12 that the set $\left[\mathbf{P}^1, \mathbf{P}^d\right]^{\mathrm{N}}$ is *a fortiori* endowed with a monoid structure, pulled back from that of $(\mathbf{N}, +)$. We denote again its law $\oplus^{\mathrm{N}}$. The statement of theorem 4.12 would be much nicer if there were an *a priori* monoid structure on $\left[\mathbf{P}^1, \mathbf{P}^d\right]^{\mathrm{N}}$ such that the degree map induces a monoid isomorphism. The author suspects that there might even be a monoid structure on the scheme $\coprod_{n \geqslant 0} \mathcal{F}_n^d$.

For the same reason as before, $\left[\mathbf{P}^1, \mathbf{P}^d\right]^{\mathbf{A}^1}$ is a group. Morel's result is the following.

***Theorem 4.14* (Morel, [9], Theorem 4.13).** — *For every integer $d \geqslant 2$, the degree map*

$$\left(\left[\mathbf{P}^1, \mathbf{P}^d\right]^{\mathbf{A}^1}, \oplus^{\mathbf{A}^1}\right) \xrightarrow{\deg} (\mathbf{Z}, +)$$

induces a group isomorphism.

Naive and $\mathbf{A}^1$-homotopy classes compare again very well.

Corollary 4.15. — *For every integer $d \geqslant 2$, the canonical map*

$$\big(\,[\mathbf{P}^1,\mathbf{P}^d]^{\mathrm{N}}, \oplus^{\mathrm{N}}\big) \longrightarrow \big(\,[\mathbf{P}^1,\mathbf{P}^d]^{\mathbf{A}^1}, \oplus^{\mathbf{A}^1}\big)$$

is a group completion.

Appendix A
Hermite inequality

The goal of this appendix is to present a proof, due to Ojanguren [**11**], of proposition 3.10 which gave a description of the naive homotopy classes of non-degenerate symmetric bilinear forms. Since the reference [**11**] is not so common in libraries, we want to include this material here. The proof is based on an elegant use of Hermite inequality for symmetric bilinear forms over the principal ring $k[T]$.

More precisely, proposition 3.10 is a consequence of the following proposition, which gives a concrete description of naive homotopies of non-degenerate symmetric bilinear forms.

Proposition A.1. — *Let n be a positive integer and let $S(T)$ be a $n\times n$ non-degenerate symmetric matrix with coefficients in $k[T]$ (that is to say an element of $\mathcal{S}_n(k[T])$). Then, there exists a matrix $P(T) \in \mathbf{SL}_n(k[T])$ such that ${}^tP(T)S(T)P(T)$ is block diagonal, where the block entries are whether units of k, whether 2×2-blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & \alpha(T)\end{bmatrix}$ with $\alpha(T) \in k[T]$.*

In particular, since we have, for any $\alpha \in k$, the equality $\begin{bmatrix} 0 & 1 \\ 1 & \alpha\end{bmatrix} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$ in $\mathrm{MW}^{\mathrm{s}}(k)$, the images of $S(0)$ and $S(1)$ in $\mathrm{MW}^{\mathrm{s}}(k)$ are equal.

A.1. Hermite inequality.
— We review first the classical version of Hermite inequality for integral symmetric bilinear forms.

***Theorem A.2* (Hermite inequality)**. — *Let n be a positive integer, L be a free $\mathbf{Z}$-module of rank n and $b : L\times L \longrightarrow \mathbf{R}$ be a symmetric bilinear form (possibly degenerate).*
Define the minimum of b to be

$$\mu(b) := \min_{x\in L-\{0\}} |b(x,x)| \qquad .$$

Then, one has the following inequality:

$$\mu(b) \leqslant \left(\frac{4}{3}\right)^{\frac{n-1}{2}} |\mathrm{D}(b)|^{\frac{1}{n}} \qquad ,$$

where $D(b)$ stands for the discriminant[8] *of the form b.*

[8] That is to say the determinant of the Gram matrix of b in any base of L.

Hermite inequality is intimately related to the fact that any real number can be approximated by an integer at distance at most $\frac{1}{2}$. It admits a generalization in the following algebraic context.

Definition A.3. — Let K be a field. An absolute value on K is a function

$$\begin{array}{ccc} K & \longrightarrow & \mathbf{R}^{+} \\ x & \mapsto & |x| \end{array}$$

satisfying the following conditions:

- $|x| = 0 \iff x = 0$;
- $\forall x, y \in K, \quad |xy| = |x||y|$;
- $|x + y| \leqslant |x| + |y|$.

The absolute value is said to be *non-archimedean* if we have furthermore

$$\forall x, y \in K, \qquad |x + y| \leqslant \max(|x|, |y|) \qquad .$$

Proposition-definition A.4. — *Let K be a field with an absolute value. A subring $R \subset K$ is of Hermite if it satisfies:*

$$\forall a \neq 0, \quad |a| \geqslant 1$$

$$\exists 0 < \rho < 1, \quad \forall x \in K, \quad \exists r \in R, \quad |x - r| \leqslant \rho$$

The following then holds:

- *A Hermite subring R is a principal ideal domain.*
- *An element $r \in R$ is a unit if and only if it satisfies $|r| = 1$.*

Definition A.5. — Let K be a field with an absolute value, $R \subset K$ be a Hermite subring, L be a free R-module of rank n and $b : L \times L \longrightarrow K$ be an R-symmetric bilinear form. Then the absolute value of the determinant of the Gram matrix of b is independent of the choice of a base. This real number is the discriminant of b and is denoted by $|\mathrm{D}(b)|$.

In this context, one can generalize Hermite inequality.

***Theorem A.6* (Generalized Hermite inequality)**. — *Let K be a field with an absolute value, $R \subset K$ be a Hermite subring L be a free R-module of rank n and $b : L \times L \longrightarrow K$ be an R-symmetric bilinear form. Define*

$$\mu(b) := \min_{x \in L - \{0\}} b(x, x) \qquad .$$

Then the following inequality holds:

$$\mu(b) \leqslant \left(\frac{1}{1 - \rho^2}\right)^{\frac{n-1}{2}} |\mathrm{D}(b)|^{\frac{1}{n}} \qquad ,$$

where ρ is the constant attached to R.
Moreover, if the absolute value of K is non-archimedean, then the inequality sharpens to:

$$\mu(b) \leqslant |\mathrm{D}(b)|^{\frac{1}{n}} \qquad .$$

Examples A.7. — 1. When $K = \mathbf{R}$ with its usual absolute value, the subring $R = \mathbf{Z}$ is of Hermite for $\rho = \frac{1}{2}$. In this case, theorem A.6 gives back the classical Hermite inequality.

2. Let $q > 1$ be a real number and k be a field. We define an absolute value on the field of rational functions $K = k(T)$ in the following way. We set $|0| = 0$; and for every non-zero rational function f, we write $f = \frac{A}{B}$ where A and B are non-zero polynomials in $k[T]$ and we set

$$|f| := q^{\deg A - \deg B} \qquad .$$

This is well defined and one checks that this absolute value is a non-archimedean. The subring $k[T] \subset K$ is of Hermite with constant $\rho = \frac{1}{q}$. Indeed, every rational function f is the sum of polynomial (its integer part) and of a rational function whose numerator is of degree strictly less than the degree of the denominator).

A.2. Proof of proposition A.1. — The proof of proposition A.1 goes by induction on the degree n.

In the case $n = 1$, the matrix $S(T)$ is constant, so there is nothing to prove.

Now on, we adopt the conventions of the example A.7 (2). Thus $K = k(T)$ is a field with a non-archimedean norm. Let $b : k[T]^n \times k[T]^n \longrightarrow k[T]$ be the $K[T]$-symmetric bilinear form associated to $S(T)$. Since $S(T)$ is non-degenerate, we have $|\mathrm{D}(b)| = 1$. The generalized Hermite inequality implies that

$$\mu(b) \leqslant 1 \qquad .$$

Since S has polynomial coefficients, $\mu(b)$ is an integer and thus only two cases are possible:

- Whether $\mu(b) = 1$. This means that there exists a vector $x \in k[T]^n$ such that $b(x,x) = \lambda \in (k[T])^\times = k^\times$. The Gram matrix of the form b in $L = \langle x\rangle \oplus \langle x\rangle^\perp$, has the following shape

$$\begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \dots & * \end{bmatrix} \qquad .$$

  Applying the induction hypothesis to the restriction of b to $\langle x\rangle^\perp$ is enough to conclude.
- Whether $\mu(b) = 0$. In this case, there exists a vector $x \in k[T]^n$ such that $b(x,x) = 0$. Up to renormalisation, we can assume that this x is indivisible. Since b is non-degenerate, there exists a vector $y \in k[T]^n$ such that $b(x,y) = 1$. The restriction of b to $\langle x, y\rangle$ is then non-degenerate: its Gram matrix in the basis $\{x, y\}$ is of the form

$$\begin{bmatrix} 0 & 1 \\ 1 & \alpha \end{bmatrix} \qquad ,$$

  with $\alpha \in k[T]$. We conclude by using the inductive hypothesis on the restriction of b to $\langle x, y\rangle^\perp$.

This concludes the proof of proposition A.1. $\square$

Appendix B
Additions of rational functions

The goal of this appendix is to compare the two addition laws on homotopy classes of endomorphisms of $\mathbf{P}^1$: the naive law denoted $\oplus^{\mathrm{N}}$ defined in §3.1 and the $\mathbf{A}^1$ law, coming from the fact that $\mathbf{P}^1$ is a suspension. More precisely, we are going to prove proposition 3.25, that is:

Proposition B.1. — *The canonical map*

$$\left(\left[\mathbf{P}^1, \mathbf{P}^1\right]^{\mathrm{N}}, \oplus^{\mathrm{N}} \right) \longrightarrow \left(\left[\mathbf{P}^1, \mathbf{P}^1\right]^{\mathbf{A}^1}, \oplus^{\mathbf{A}^1} \right)$$

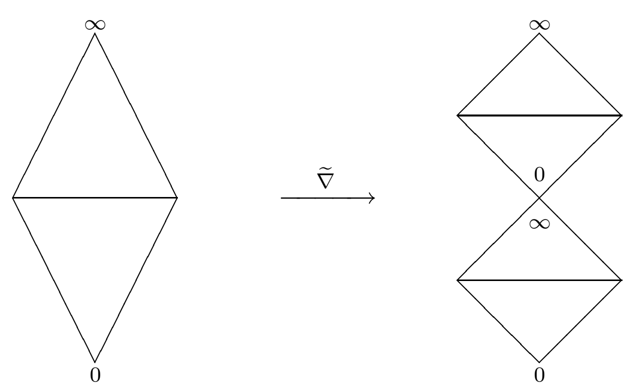
is a monoid morphism.

Proof. — Let g_1 and g_2 be two pointed rational functions. We need to prove that the image of the rational function $g_1 \oplus^{\mathrm{N}} g_2$ in $[\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}$ is equal to $g_1 \oplus^{\mathbf{A}^1} g_2$. Since the monoid $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$ of naive homotopy classes of rational functions is generated by its elements of degree 1 (*c.f.* lemma 3.14), it is enough to deal with the case when g_1 is of degree 1. Up to homotopy, one can even assume that g_1 is of the form $\frac{X}{a}$ for some $a \in k^{\times}$. For g_1 of this form, the formula for the $\oplus^{\mathrm{N}}$-sum is

$$\frac{X}{a} \oplus^{\mathrm{N}} g_2 = \frac{X}{a} - \frac{1}{a^2}\frac{1}{g_2} \qquad .$$

Definition B.2. — Let $\mathbf{P}^1 \coprod_{0\sim\infty} \mathbf{P}^1$ be the cofiber of the map $\mathrm{S}^0 = \{0, \infty\} \hookrightarrow \mathbf{P}^1 \coprod \mathbf{P}^1$. (Equivalently, $\mathbf{P}^1 \coprod_{0\sim\infty} \mathbf{P}^1$ is the union of two copies of $\mathbf{P}^1$ with the point 0 in the first copy identified with the point ∞ in the second one). The base point is taken at ∞. As $\mathbf{P}^1$ is up to homotopy an *unreduced* suspension, there is a canonical map[(9)] in the $\mathbf{A}^1$-homotopy category:

$$\mathbf{P}^1 \quad \xrightarrow{\tilde{\nabla}} \quad \mathbf{P}^1 \coprod_{0\sim\infty} \mathbf{P}^1$$



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(9) An explicit model for this map is for example the cofiber map $\mathbf{P}^1 \longrightarrow \mathbf{P}^1 \big/ \left(\mathbf{P}^1 - \{0, \infty\}\right)$.

Lemma B.3. — *Let $g : \mathbf{P}^1 \longrightarrow \mathbf{P}^1$ be a pointed rational function and let f be the pointed rational function $f = \frac{X}{a} - \frac{1}{g}$. One has $f^{-1}(\infty) = \{\infty\} \coprod g^{-1}(0)$ and we denote $\overline{f}$ the induced map between the cofibers:*

$$\overline{f} : \mathbf{P}^1 \big/ \mathbf{P}^1 - \big(\{\infty\} \coprod g^{-1}(0)\big) \longrightarrow \mathbf{P}^1 \big/ \mathbf{P}^1 - \{\infty\} \qquad .$$

*Then the following diagram of spaces (in the sense of Morel and Voevodsky [**10**])*

$$\begin{tikzcd}
\mathbf{P}^1 \arrow[rr, "f"] \arrow[dr] \arrow[ddd, "\widetilde{\nabla}"'] & & \mathbf{P}^1 \arrow[d, "\approx"] \\
& \mathbf{P}^1 \big/ \mathbf{P}^1 - \big(\{\infty\} \coprod g^{-1}(0)\big) \arrow[r, "\overline{f}"] \arrow[d, equal] & \mathbf{P}^1 \big/ \mathbf{P}^1 - \{\infty\} \\
& \Big(\mathbf{P}^1 \big/ \mathbf{P}^1 - \{\infty\}\Big) \coprod_{* \sim *} \Big(\mathbf{P}^1 \big/ \mathbf{P}^1 - g^{-1}(0)\Big) & \\
\mathbf{P}^1 \coprod_{0 \sim \infty} \mathbf{P}^1 \arrow[ur] \arrow[rr, "\frac{X}{a} \coprod_{0\sim\infty} -\frac{1}{g}"'] & & \mathbf{P}^1 \arrow[uu, "\approx"']
\end{tikzcd}$$

homotopy commutes.

Proof. — The main point is to prove that the two following diagrams:

$$\begin{tikzcd}
\mathbf{P}^1 \arrow[rr, "\frac{X}{a}"] \arrow[d] & & \mathbf{P}^1 \arrow[d] \\
\mathbf{P}^1 \big/ \mathbf{P}^1 - \{\infty\} \arrow[r, hook] & \mathbf{P}^1 \big/ \mathbf{P}^1 - \big(\{\infty\} \coprod g^{-1}(0)\big) \arrow[r, "\overline{f}"] & \mathbf{P}^1 \big/ \mathbf{P}^1 - \{\infty\}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{P}^1 \arrow[rr, "g"] \arrow[d] & & \mathbf{P}^1 \arrow[d] \\
\mathbf{P}^1 \big/ \mathbf{P}^1 - g^{-1}(0) \arrow[r, hook] & \mathbf{P}^1 \big/ \mathbf{P}^1 - \big(\{\infty\} \coprod g^{-1}(0)\big) \arrow[r, "\overline{f}"] & \mathbf{P}^1 \big/ \mathbf{P}^1 - \{\infty\}
\end{tikzcd}$$

homotopy commute. The proof is the same in both cases, so we give the details only for the last diagram.

According to the homotopy purity theorem of Morel and Voevodsky (*c.f.* [**10**], theorem 2.23, p. 115), the space $\mathbf{P}^1 \big/ \mathbf{P}^1 - g^{-1}(0)$ is homotopy equivalent to the Thom space of the normal bundle

of the closed immersion $g^{-1}(0) \hookrightarrow \mathbf{P}^1$. Since $g^{-1}(0) \cap \{\infty\} = \emptyset$, one has a homotopy equivalence $\mathbf{A}^1 / \mathbf{A}^1 - g^{-1}(0) \xrightarrow{\approx} \mathbf{P}^1 / \mathbf{P}^1 - g^{-1}(0)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{A}^1 \arrow[rr, "f_{|\mathbf{A}^1}"] \arrow[d] & & \mathbf{P}^1 \arrow[d] \\
\mathbf{A}^1 / \mathbf{A}^1 - g^{-1}(0) \arrow[rr, "\overline{f_{|\mathbf{A}^1}}"] \arrow[d, "\approx"'] & & \mathbf{P}^1 / \mathbf{P}^1 - \{\infty\} \\
\mathbf{P}^1 / \mathbf{P}^1 - g^{-1}(0) \arrow[rru, "\overline{f}"'] & &
\end{tikzcd} .$$

But now, there is a naive homotopy of morphisms of pairs:

$$\begin{array}{ccc}
\left(\mathbf{A}^1, \mathbf{A}^1 - g^{-1}(0)\right) \times \mathbf{A}^1 & \longrightarrow & \left(\mathbf{P}^1, \mathbf{P}^1 - \{\infty\}\right) \\
(X, \quad T) & \mapsto & T\frac{X}{a} - \frac{1}{g(X)}
\end{array}$$

between $f_{|\mathbf{A}^1}$ and $(-\frac{1}{g})_{|\mathbf{A}^1}$, which implies the homotopy commutativity of the diagram considered. This concludes the proof of the lemma. $\square$

By the previous lemma, to prove proposition B.1 one has to express the composite map

$$\mathbf{P}^1 \xrightarrow{\tilde{\nabla}} \mathbf{P}^1 \coprod_{0\sim\infty} \mathbf{P}^1 \xrightarrow{\frac{X}{a} \coprod_{0\sim\infty} -\frac{1}{g}} \mathbf{P}^1$$

as a sum in the group $[\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}$. The result is given in the next lemma, which is stated in the following context:

- Let Y be the non-reduced suspension of a pointed space. The base point in Y is taken at ∞; the other distinguished point is denoted 0.
- Let $(Z; z_0)$ be an $\mathbf{A}^1$-connected pointed space.
- Let $g_1, g_2 : Y \longrightarrow Z$ be two maps such that $g_1(0) = g_2(\infty)$. We assume that g_1 is pointed, that is to say that $g_1(\infty) = z_0$.
- Let f be the pointed map $(g_1 \coprod_{0\sim\infty} g_2) \circ \tilde{\nabla} : Y \longrightarrow Z$.
- Let $H : Y \times \mathbf{A}^1 \longrightarrow Z$ be a naive homotopy from g_2 to a pointed map, say $g_3 : Y \longrightarrow Z$.
- Let $\alpha : \mathbf{A}^1 \longrightarrow Z$ be the path $T \mapsto H(\infty, T)$ in Z from $g_2(\infty)$ to z_0.
- Let $\beta : \mathbf{A}^1 \longrightarrow Z$ be the image through g_1 of the canonical path in Y from ∞ to 0; β is thus a path in Z from z_0 to $g_1(0) = g_2(\infty)$.
- Let γ be the concatenation of the paths β and α; γ is thus a loop in Z at z_0.

Lemma B.4. — *In the group $[Y, Z]^{\mathbf{A}^1}$, one has the identity:*

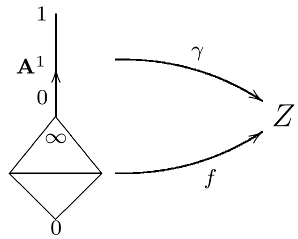
$$\gamma \cdot f = (\gamma \cdot g_1) \oplus^{\mathbf{A}^1} g_3 \qquad .$$

Above here, the dots "$\cdot$" denote the action of an element of $\pi_1^{\mathbf{A}^1}(Z;z_0)$ *on* $[Y,Z]^{\mathbf{A}^1}$.

Proof. — It is a consequence of the following facts:

– Up to homotopy, Y can be replaced by $\mathbf{A}^1 \coprod_{0\sim\infty} Y$, pointed at $1 \in \mathbf{A}^1$. The element $\gamma \cdot f$ is represented by the map

$$\mathbf{A}^1 \coprod_{0\sim\infty} Y \xrightarrow{\gamma \coprod_{0\sim\infty} f} Z$$



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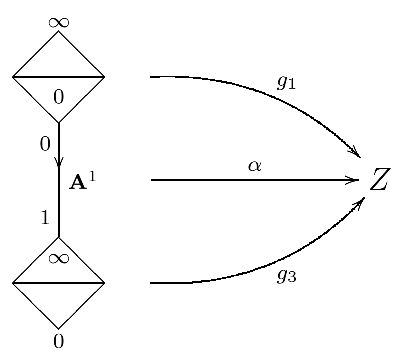
– Up to homotopy, $Y \coprod_{0\sim\infty} Y$ can be replaced by $Y \coprod_{0\sim 0} \mathbf{A}^1 \coprod_{1\sim\infty} Y$, the map $Y \coprod_{0\sim\infty} Y \xrightarrow{g_1 \coprod_{0\sim\infty} g_2} Z$ is then homotopic to the map

$$Y \coprod_{0\sim 0} \mathbf{A}^1 \coprod_{1\sim\infty} Y \xrightarrow{g_1 \coprod_{0\sim 0} \alpha \coprod_{1\sim\infty} g_3} Z$$



– The restriction of the map

$$\mathbf{A}^1 \coprod_{0\sim\infty} Y \coprod_{0\sim 0} \mathbf{A}^1 \coprod_{1\sim\infty} Y \xrightarrow{\gamma \coprod_{0\sim\infty} f \coprod_{0\sim 0} \alpha \coprod_{1\sim\infty} g_3} Z$$

to $\mathbf{A}^1 \coprod_{0\sim 0} \mathbf{A}^1 \coprod_{1\sim 0} \mathbf{A}^1$ (the $\mathbf{A}^1$ in the middle is the domain of β) is the concatenation of γ^{-1} and γ and is thus null-homotopic. So, up to homotopy, the map $\gamma \coprod_{0\sim\infty} f \coprod_{0\sim 0} \alpha \coprod_{1\sim\infty} g_3$ factors through the cofiber $\left(\mathbf{A}^1 \coprod_{0\sim\infty} Y \coprod_{0\sim 0} \mathbf{A}^1 \coprod_{1\sim\infty} Y\right) \big/ \left(\mathbf{A}^1 \coprod_{0\sim 0} \mathbf{A}^1 \coprod_{1\sim 0} \mathbf{A}^1\right) \approx Y \vee Y$.

– The following composite map

$$\begin{tikzcd}
Y \arrow[r, "\approx"] \arrow[d, dashed] & \mathbf{A}^1 \coprod\limits_{0\sim\infty} Y \arrow[r, "\mathrm{id} \coprod\limits_{0\sim\infty} \widetilde{\nabla}"] & \mathbf{A}^1 \coprod\limits_{0\sim\infty} Y \coprod\limits_{0\sim\infty} Y \arrow[d, "\approx"] \\
Y \vee Y & \left(\mathbf{A}^1 \coprod\limits_{0\sim\infty} Y \coprod\limits_{0\sim 0} \mathbf{A}^1 \coprod\limits_{1\sim\infty} Y\right) \Big/ \left(\mathbf{A}^1 \coprod\limits_{0\sim 0} \mathbf{A}^1 \coprod\limits_{1\sim 0} \mathbf{A}^1\right) \arrow[l, "\approx"'] & \mathbf{A}^1 \coprod\limits_{0\sim\infty} Y \coprod\limits_{0\sim 0} \mathbf{A}^1 \coprod\limits_{1\sim\infty} Y \arrow[l]
\end{tikzcd}$$

is a model for the co-diagonal $Y \xrightarrow{\nabla} Y \vee Y$.

$\square$

In our situation, there is a "universal" homotopy between $-\frac{1}{g}$ and g, which is given by composition at the target of g with a naive homotopy between between $-\frac{1}{X}$ and X. One can choose

$$\frac{(-T^2+2T)X-(T^3-3T^2+T+1)}{(-T+1)X+(-T^2+2T)}$$

as an example of such a homotopy.

Lemma B.5. — *Let $\alpha : \mathbf{A}^1 \longrightarrow \mathbf{P}^1$ be the path $T \mapsto [T : 1-T^2]$, $\beta : \mathbf{A}^1 \longrightarrow \mathbf{P}^1$ be the path $T \mapsto [1-T : aT]$ and γ be the loop given by concatenation of α and β. Then for any pointed rational function $F : \mathbf{P}^1 \longrightarrow \mathbf{P}^1$, one has the identity*

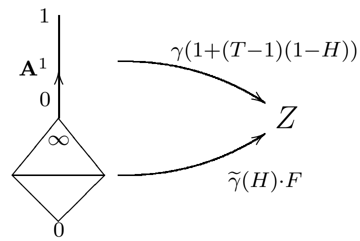
$$\gamma \cdot F = a^2 F$$

in $[\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}$.

Proof. — The projective line $\mathbf{P}^1$ is homotopy equivalent to the homogeneous space $\mathbf{SL}_2 / \mathbf{G_m}$. So any path to $\mathbf{P}^1$ lifts to a path in $\mathbf{SL}_2$. Moreover, up to homotopy, one can use the product in $\mathbf{SL}_2$ to compose paths. More precisely, let $\widetilde{\beta} : \mathbf{A}^1 \longrightarrow \mathbf{SL}_2$ be a lift of β such that $\widetilde{\beta}(0) = \mathrm{id}$ and let $\widetilde{\alpha} : \mathbf{A}^1 \longrightarrow \mathbf{SL}_2$ be a lift of α such that $\widetilde{\alpha}(0) = \widetilde{\beta}(1)$. Then a lift $\widetilde{\gamma}$ of the loop γ is

$$\begin{array}{rccc} \widetilde{\gamma} : & \mathbf{A}^1 & \longrightarrow & \mathbf{SL}_2 \\ & T & \mapsto & \widetilde{\beta}(T) \cdot \widetilde{\beta}(1)^{-1} \cdot \widetilde{\alpha}(T) \end{array}$$

Let $c_\infty : \mathbf{A}^1 \longrightarrow \mathbf{P}^1$ be the constant path at $\infty \in \mathbf{P}^1$. The action of $\mathbf{SL}_2$ on $\mathbf{P}^1$ can be used to define a naive homotopy between $\gamma \coprod_{0\sim\infty} F$ and $c_\infty \coprod_{0\sim\infty} (\widetilde{\alpha}(1)\cdot F)$, namely[(10)]



.

Explicitly, one can take $\widetilde{\alpha}(T) = \begin{bmatrix} a(-T^2+2T) & (-2+\frac{1}{a})T^2 + (4-\frac{1}{a})T - \frac{1}{a} \\ a(-T+1) & (-2+\frac{1}{a})T+2 \end{bmatrix}$ and $\widetilde{\beta}(T) = \begin{bmatrix} 1-T & -\frac{T}{a} \\ aT & 1+T \end{bmatrix}$. So $\widetilde{\gamma}(1) = \widetilde{\alpha}(1) = \begin{bmatrix} a & 2-\frac{1}{a} \\ 0 & \frac{1}{a} \end{bmatrix}$. The result follows since $\widetilde{\gamma}(1)\cdot F = a^2F + 2 - \frac{1}{a}$ is canonically homotopic to a^2F. □

Lemma B.4 and lemma B.5 together imply proposition B.1. Indeed, for every $a \in k^\times$ and for every pointed rational function g one has the identity in $[\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}$

$$aX \oplus^{\mathrm{N}} g = aX - \frac{a^2}{g} = a^2\Big(\frac{X}{a} - \frac{1}{g}\Big) = aX \oplus^{\mathbf{A}^1} g \qquad .$$

□

Remark B.6. — It is likely that the preceding method could be refined to show that for every positive integer n the diagram of spaces

$$\begin{tikzcd} \mathcal{F}_1 \times \mathcal{F}_n \arrow[r, "\oplus^{\mathrm{N}}"] \arrow[d, hook] & \mathcal{F}_{n+1} \arrow[d, hook] \\ \Omega^{\mathbf{P}^1}_1 \mathbf{P}^1 \times \Omega^{\mathbf{P}^1}_n \mathbf{P}^1 \arrow[r, "\oplus^{\mathbf{A}^1}"] & \Omega^{\mathbf{P}^1}_{n+1} \mathbf{P}^1 \end{tikzcd}$$

homotopy commutes.

Appendix C
Hankel matrices

Let n be a positive integer. We described in §3.2 a way to produce a non-degenerate $n\times n$ symmetric matrix out of a pointed rational function. However, for dimension reasons, it appears that for $n \geqslant 3$, not any non-degenerate $n\times n$ symmetric matrix is the Bézout form of some rational function. Indeed, one observes (*c.f.* lemma C.2) that the inverse of the Bézout form of rational function is *Hankel*, that is has constant value along anti-diagonals, *c.f.* definition C.1. It turns out that this necessary condition is also sufficient. The goal of this appendix is to review this old story with our conventions.

[(10)]Below, $H \in \mathbf{A}^1$ is the parameter of the homotopy and $T \in \mathbf{A}^1$ belongs to the domain of γ.

Definition C.1. — A symmetric matrix S is of Hankel if the value of its entries $s_{p,q}$ depends only on $p+q$. For every integer n, let $\mathcal{H}_n$ be the scheme of non-degenerate $n\times n$ Hankel matrices; $\mathcal{H}_n$ is thus a closed subscheme of $\mathcal{S}_n$ of dimension $2n-1$.

Lemma C.2. — *Let R be a ring, n be a positive integer and $f=\frac{A}{B}\in\mathcal{F}_n(R)$ be a rational function. Then the matrix* $\mathrm{Béz}_n(f)^{-1}$ *is Hankel.*

Proof. — Consider the universal case when the coefficients of A and B are indeterminates, and let K be the algebraic closure of the quotient field of $R=\mathbf{Z}[a_i,b_j]$.
The polynomial A is split in K. Let $(\alpha_i)_{1\leqslant i\leqslant n}$ be the roots of A in K and let E be the étale K-algebra $K[X]\big/(A)$. We are going to show that $\mathrm{Béz}_n(f)^{-1}$ is the matrix in the canonical basis $(1,X,\dots,X^{n-1})$ of E, of the following trace form

$$\begin{array}{ccc} E\times E & \longrightarrow & K \\ (P,Q) & \mapsto & \mathrm{tr}_{E/K}\left(\frac{PQ}{A'B}\right) \end{array}$$

The generic entry of this matrix is $\mathrm{tr}_{E/K}\left(\frac{X^{p+q}}{A'B}\right)$ which proves that $\mathrm{Béz}_n(f)^{-1}$ is indeed Hankel.

Let V be the Vandermonde matrix associated to the roots of A, that is

$$V:=\begin{bmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \dots & \dots & \dots & \dots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{bmatrix}\qquad .$$

By construction of $\mathrm{Béz}_n(A,B)$, we have the following identity

$${}^tV\cdot\mathrm{Béz}_n(A,B)\cdot V=\left[\delta_{A,B}(\alpha_p,\alpha_q)\right]\qquad ,$$

where $\delta_{A,B}$ was introduced in the definition of $\mathrm{Béz}_n(A,B)$. For every integers $1\leqslant p,q\leqslant n$, one has

$$\delta_{A,B}(\alpha_p,\alpha_q)=\begin{cases} A'(\alpha_p)B(\alpha_p) & \text{whenever } p=q \\ 0 & \text{otherwise} \end{cases}\qquad .$$

This shows that $\mathrm{Béz}_n(f)^{-1}$ is indeed the matrix of the previous trace form. □

Remark C.3. — We keep the notations introduced in the proof of lemma C.2. Let also ρ be the linear form *residue*

$$\begin{array}{cccc} \rho: & E & \longrightarrow & K \\ & X^i & \mapsto & \begin{cases} 1 & \text{if } i=n-1 \\ 0 & \text{otherwise} \end{cases} \end{array}$$

By a classical identity due to Euler (see [**13**], chapitre 2, n°6, lemme 2 for example), we have

$$\forall P\in E,\qquad \rho(P)=\mathrm{tr}_{E/K}\left(\frac{P}{A'}\right)\qquad .$$

The matrix $\text{Béz}_n(A,B)^{-1}$ is thus the matrix of the symmetric bilinear form $(P,Q) \mapsto \rho\left(\frac{PQ}{B}\right) = \rho(PQV)$, where V is a polynomial associated to a Bézout relation $AU + BV = 1$.
One thus deduces the following algorithm to compute the Hankel matrix $\text{Béz}_n(A,B)^{-1}$. One formally expands in $R[X, X^{-1}]]$ the rational function $\frac{V}{A}$:

$$\frac{V}{A} = s_1 X^{-1} + s_2 X^{-2} + \cdots + s_{2n-1} X^{-(2n-1)} + s_{2n} X^{-2n} + \mathrm{O}(X^{-(2n+1)}) \quad . \tag{C.3}$$

The matrix $\text{Béz}_n(A,B)^{-1}$ is then given by

$$\text{Béz}_n(A,B)^{-1} = \begin{bmatrix} s_1 & s_2 & s_3 & \dots & s_n \\ s_2 & s_3 & \dots & \dots & s_{n+1} \\ s_3 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ s_n & s_{n+1} & \dots & \dots & s_{2n-1} \end{bmatrix} \quad .$$

Definition C.4. — Let R be a ring and let n be a positive integer.

- For all $(A,B) \in \mathcal{F}_n(R)$, there exists a unique pair of polynomials (U_1, V_1) with $\deg U_1 = n-1$, $\deg V_1 \leqslant n-1$ and such that
$$AU_1 + BV_1 = X^{2n-1} \quad .$$
One defines a scheme morphism $\varphi_n : \mathcal{F}_n \longrightarrow \mathbf{A}^1$ by associating to the pair (A,B) the opposite of the coefficient of X^{n-1} in V_1. (Note that the scalar $\varphi_n(A,B)$ just defined is also equal to the opposite of the coefficient s_{2n} of X^{-2n} in the formal expansion (C.3) of $\frac{V}{A}$).
- Let Hank_n be the morphism of schemes
$$\begin{array}{rccc} \text{Hank}_n : & \mathcal{F}_n & \longrightarrow & \mathcal{H}_n \\ & (A,B) & \mapsto & \text{Béz}_n(A,B)^{-1} \end{array}$$

Proposition C.5. — *For every integer n, the morphism*

$$\mathcal{F}_n \xrightarrow{\text{Hank}_n \times \varphi_n} \mathcal{H}_n \times \mathbf{A}^1$$

is a $\mathbf{G_a}$-equivariant isomorphism of schemes.

Proof. — The morphism was checked to be equivariant in §3.4.3. Moreover, remark C.3 suggests a way to produce the inverse morphism. Let

$$\begin{array}{rccc} \psi_n : & \mathcal{H}_n \times \mathbf{A}^1 & \longrightarrow & \mathcal{F}_n \\ & (H,v) & \mapsto & (A,B) \end{array}$$

where A is the degree n monic polynomial whose coefficients a_i are given by

$$\begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_{n-2} \\ a_{n-1} \end{bmatrix} = H^{-1} \begin{bmatrix} -s_{n+1} \\ -s_{n+2} \\ \dots \\ -s_{2n-1} \\ v \end{bmatrix} \quad ,$$

and where B is the unique polynomial of degree $\leqslant n-1$ given by the Bézout relation $AU+BV=1$ for $V=\mathrm{E}((\sum_{i=1}^{n} s_i X^{-i})A)$, E denoting the integer part.
By construction, ψ_n is inverse to $\mathrm{Hank}_n \times \varphi_n$. □

Remark C.6. — The scheme $\mathcal{F}_n$ is a $\mathbf{G_a}$-torsor over the base $\mathcal{H}_n$, which is affine. Thus it is not surprising that this torsor is trivial.

Remark C.7. — Proposition C.5 allows to reformulate our results in terms of Hankel matrices. For example, there is a canonical monoid structure denoted $\oplus^{\mathrm{N}}$ on the set $\coprod_{n\geqslant 0}(\pi_0^{\mathrm{N}}\mathcal{H}_n)(k)$ such that the map

$$\begin{array}{ccc} \mathcal{H}_n & \longrightarrow & \mathcal{S}_n \\ H & \mapsto & H^{-1} \end{array}$$

induces a monoid isomorphism

$$\Big(\coprod_{n\geqslant 0}(\pi_0^{\mathrm{N}}\mathcal{H}_n)(k),\oplus^{\mathrm{N}}\Big) \xrightarrow{\simeq} \Big(\coprod_{n\geqslant 0}(\pi_0^{\mathrm{N}}\mathcal{S}_n)(k),\oplus\Big) \qquad .$$

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